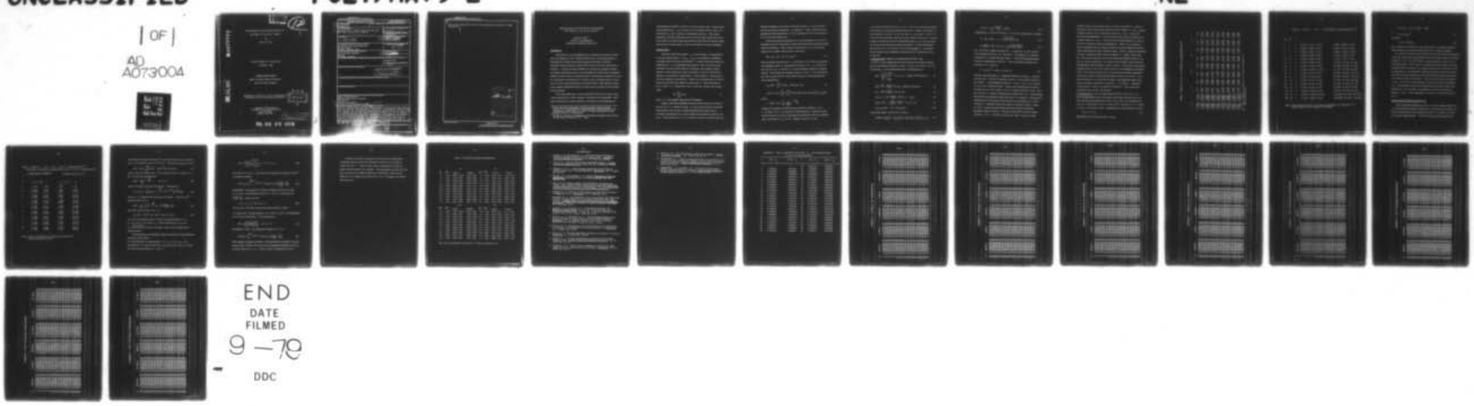
AD-A073 004 POLYTECHNIC INST OF NEW YORK BROOKLYN DEPT OF MATHEMATICS F/G 9/1
APPROXIMATIONS OF NORMAL RANGE REVISITED WITH NEW TABLES FOR D2--ETC(U)
JAN 79 J H KAO N00014-77-C-0601

UNCLASSIFIED POLY/MA79-2 NL

| OF |
AD
A073004



END
DATE
FILMED
9 -79
DDC

Approximations of Normal Range Revised with

New Tables for d_2 and d_3 - Factors

by

John H. K. Kao

Technical Report No. Poly/MA 79-2

19 January, 1979

PREPARED UNDER CONTRACT

N00014-77-C-0601 PROJECT NR 042-377

OFFICE OF NAVAL RESEARCH

Department of Mathematics
Polytechnic Institute of New York
333 Jay Street
Brooklyn, New York

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

(14)

1. REPORT NUMBER
POLY/MA79-2

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

(6)

4. TITLE (and Subtitle)
Approximations of Normal Range Revisited with New Tables for d_2 and d_3 - Factors.

(9)

5. TYPE OF REPORT & PERIOD COVERED
Technical rept.

6. PERFORMING ORG. REPORT NUMBER
POLY/MA79-2

(10)

7. AUTHOR(s)
John H. K. Kao

(15)

8. CONTRACT OR GRANT NUMBER(s)
N00014-77-C-0601
Project NR042-377

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Polytechnic Institute of New York
333 Jay Street
Brooklyn, New York 11201

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
(12) 28p.

11. CONTROLLING OFFICE NAME AND ADDRESS
Office of Naval Research
Code 436
Arlington, Va 22217

(11)

12. REPORT DATE
19 Jan[redacted]79

13. NUMBER OF PAGES
25

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Quality Control Charts
Sampling Inspection by Variables
MIL-STD-414
Cumulative Sum Charts
Analysis of Variance tests by range.

-squared

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Patnaik's X-approximation and Cox's X^2-approximation to the normal mean range are revisited and many inaccurate entries discovered. Investigations revealed, among other reasons, that the original Tippett's calculations on $\hat{d_2}$ and $\hat{d_3}$ -factors were of inadequate accuracy. New tables for $\hat{d_2}$ and $\hat{d_3}$ -factors as well as constants accociated with Patnaik's and Cox's approximations are recomputed and tabulated. These new tables will be useful for revising the MIL-STD-414 range plans for inspection by variable and for the analysis of variance tests when ranges or mean ranges are used in place of sample variances or standard deviations, as

Next page

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

411 319

mt

well as for constructing limits for the cumulative sum charts for range variability.

Accession For	
NTIS GRA&I	☑
DDC TAB	☐
Unannounced	☐
Justification Letter on file	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or special
A	

Approximations of Normal Range Revisited With New Tables for d_2 and d_3 Factors*

John H. K. Kao**
Division of Management
Polytechnic Institute of New York

Introduction

MIL-STD-414, the sampling plans by variables has not been revised since its inception in 1957, while MIL-STD-105 D, the sampling plans by attributes has reached international popularity through 4 revisions. Recently revision activities on MIL-STD-414 have received a great deal of attention. A British proposal BS6002 [16] restored the graphical method for double-specification-limit case and omitted the range method completely. BS6002 was sent to ISO for draft discussion. The American position is to keep the range plans and furthermore to devise a set of variables plans to match the OC curves of plans published in MIL-STD-105 D. This position created some technical difficulties which are under careful scrutiny by both industry and government.

If the range tables (pp. 61-84 of MIL-STD-414) are to be kept, much of the entries(close to 9000 in number) will have to be recomputed. This report indicates the reasons why this is necessary by revisiting the two

---

* Work supported by the Office of Naval Research under Contract N00014-77-C-0601/NR042-377 with Polytechnic Institute of New York. Reproduction of this report in whole or in part is permitted for any purpose of the U. S. Government.

** Professor, Division of Management, Polytechnic Institute of New York, 333 Jay Street, Brooklyn, N. Y. 11201.

approximations (Patnaik's and Cox's) for the normal range. Chiefly, these approximations gave rise to tables of four constants whose values were based upon d_2 and d_3-factors of inadequate accuracy. This report publishes in its three appendices more accurate values which will be useful not only in revising range table entries in MIL-STD-414 but also useful wherever the normal range is used as an estimate of normal variability.

Sample Range

When the observation number n is not too large, it is appropriate to use the sample range, $R = X_{[n]} - X_{[1]}$ as an estimate of population variability under the normal assumption. In control chart work where the subgroup size n is intentionally kept small (in order to avoid possible inclusion of process shifts within the subgroup), the situation is ideal. However, if the sample size needs to be larger, such as in sampling inspection, a great deal of "information" is lost by using R which ignores all but two extreme observations. Under such circumstances it is wise to break up, if possible, the sample at random into m small subgroups of size n each, i.e., sample size = (m x n) and use, instead, the average range

$$\overline{R} = \sum_{i=1}^{m} R_i/m \qquad (1)$$

where R_i is the sample range of the i^{th} subgroup.

Clearly, with fixed available resources when observing a sample of size (m x n), if n chosen is small, m will be large and the resultant $\overline{R}$ would reflect a loss of inter-subgroup information through massive averaging, although the loss of intra-subgroup information is kept a a low level. On the other hand, if m is reduced, in order to minimize the loss

through averaging, a price has to be paid for larger n which results in loss of intra-group information. An optimum n exists, which fact was studied by Grubbs and Weaver [1] and confirmed by Cox [2] and found to be 7 or 8 for the normal case.

Under the normal assumption for X, the distribution of R and its moments can not be expressed in closed form and hence numerical integrations are necessary. This resulted in the famous d_2 and d_3- factors widely used in quality control charts, which are:

$$ER = d_2\sigma' \quad \text{and} \quad \mathrm{Var}\ R = (d_3\sigma')^2 . \tag{2}$$

Here the quality characteristic X is assumed to be normally distributed with population mean $\bar{x}'$ and population variance σ'^2, i.e., $N(X\,|\,\bar{x}', \sigma')$ and $N(Y\,|\,0, 1)$ *for* $Y = (X-\bar{x}')/\sigma'$ to use the standard notations of quality control. If one defines the standard range $W = Y[n] - Y[1]$, then $W=R/\sigma'$ and $EW = d_2$ and $\mathrm{Var}\ W = (d_3)^2$ which are given by the following well-known expressions:

$$d_2 = EW = \int_{-\infty}^{+\infty} [1-F^n(y) - Q^n(y)]\, dy \quad \text{and} \tag{3}$$

$$(d_3)^2 = \mathrm{Var}\, W = 2\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \left\{1-F^n(y)-Q^n(z)+[F(y)-F(z)]^n\right\} dz-(d_2)^2 \tag{4}$$

where

$$1 - Q(y) = F(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt.$$

Tippett [3] using Eq. (3) with Gaussian quadrature obtained d_2 for $n = 2(1)1000$ in 5 D. (5D means five decimal places.) Instead of using Eq. (4) directly, he chose an approximate distribution of $\bar{W}$ and computed only a few values for d_3 in 4 D. Tippett's values of d_2 for

$n = 2(1)\ 500\ (10)\ 1000$ in 5D and d_3 for $n = 2(1)\ 20$ in 4D can also be found in Biometrika [4] Tables 27 and 20 respectively. Grubbs and Weaver [1] also tabulated 11 pairs of d_2 and d_3 (their d_n and k_n) for $n = 2(1)\ 12$, in 4D and 5D respectively, but some of their values are in error. In need of better accuracy we [5] have recomputed values of d_2 and d_3 by Eqs. (3) and (4) above directly by numerical methods. We herewith present these factors for $n = 2(1)\ 50$ in 7 D and 8 D in Appendix A. Our values check favorably with computations calculated from Teichroew [15] for $n = 2(1)\ 20$.

χ-approximation Patnaik [6] approximated $\overline{W} = \overline{R}/\sigma'$ with $c\chi/\sqrt{\nu} = c'\chi$ by equating the first two moments of the latter with the former and obtained 40 pairs of c and ν for $m = 1(1)\ 5$ and $n = 3(1)\ 10$. The density χ can be easily found from χ^2-distribution as

$$f(\chi) = \frac{\chi^{\nu-1} e^{-\chi^2/2}}{2^{\frac{\nu}{2}-1}\,\Gamma(\frac{\nu}{2})}, \quad \nu > 0,\ \chi > 0, \text{ whose } k^{th} \text{ moment is,} \tag{5}$$

$$E\chi^k = 2^{\frac{k}{2}}\,\Gamma\left(\frac{\nu+k}{2}\right)/\,\Gamma(\nu/2), \text{ which in turn gives,} \tag{6}$$

$$E\chi = \sqrt{2}\,\Gamma\left(\frac{\nu+k}{2}\right)/\,\Gamma(\nu/2) \text{ and} \tag{7}$$

$$\mathrm{Var}\,\chi = \nu - 2\,\Gamma^2\left(\frac{\nu+1}{2}\right)/\,\Gamma^2(\nu/2), \quad \text{hence} \tag{8}$$

$$E(c\chi/\sqrt{\nu}) = c\sqrt{\frac{2}{\nu}}\,\Gamma\left(\frac{\nu+1}{2}\right)/\,\Gamma(\nu/2) \text{ and} \tag{9}$$

$$\mathrm{Var}\,(c\chi/\sqrt{\nu}) = c^2 - E^2(c\chi/\sqrt{\nu}) \tag{10}$$

On the other hand, from Eq. (2), we have,

$$E(\overline{R}/\sigma') = \frac{1}{m}\sum R/\sigma' = \frac{1}{m}\sum ER/\sigma' = \frac{1}{m}\sum ER/\sigma' = \frac{1}{m}\sum d_2 = d_2 \tag{11}$$

$$V(\bar{R}/\sigma') = \frac{1}{\sigma'^2} \frac{(d_3\sigma')^2}{m} = d_3^2/m . \tag{12}$$

Equating Eqs. (9) and (10) with Eqs. (11) and (12) respectively, we obtain

$$c^2 - d_2^2 = d_3^2/m, \quad \text{or } c = \sqrt{d_2^2 + d_3^2/m} \quad \text{and} \tag{13}$$

$$\Gamma\left(\frac{\nu+1}{2}\right) / \sqrt{\nu}\ \Gamma(\nu/2) = d_2/\sqrt{2(d_2^2 + d_3^2/m)} . \tag{14}$$

Eq. (13) gives an explicit solution for c in terms of d_2 and d_3 and ν can be obtained from Eq. (14) iteratively. But Patnaik [6] and later Resnikoff [7], used the Stirling's large ν formula for gamma function and gave an approximate value of ν from solving the following cubic equation, for $a = d_2m/d_3$,

$$\nu^3 = (a/2)\nu^2 - (a/8)\nu + (a/16) = 0 , \tag{15}$$

and then solve the value of c afterwards (Instead of solving c directly from Eq. (13)). David [11] used Eq. (13) for c but a similar scheme to Eq. (15) for ν and presented 54 pairs of c and ν for $m = 1(1)\ 5$ and 10, $n = 2(1)\ 10$ (cf. Table 30 of [4]). Duncan [12] extended David's table to 210 pairs for $m = 1(1)\ 15$ and $n = 2(1)\ 15$. Duncan renamed c by d_2^* , since $d_2^* \to d_2$ as $m \to \infty$ which is evident from looking at Eq. (13). Unfortunately, both David's and Duncan's values have only 1 to 2 decimals. Thomson [13] interpolated the Γ-function at close intervals and tabulated for $n = 2(10)$ 10 and $m = 1$, yielding 9 pairs (See values in brackets in Table 1 below) of ν and c which are much better than those obtained by Patnaik, Resnikoff, David and Duncan. The facts remain that in order to obtain accurate c and ν from Eqs. (13) and (14); better d_2 and d_3 than

Tippett's values are necessary (See our values in Appendix A). Patnaik, Resnikoff, David, Duncan and Thomson all used Tippett's d_2 and d_3 values [Tables 27 and 20, Ref. 4] and their results on c and ν check each other only approximately. It is the purpose of this report to publish more accurate values of c and ν from our more accurate values of d_2 and d_3 given by our previous research (Appendix A). Table 1 is a comparison between Patnaik's and our results rounded to one more digit than Patnaik's values indicating many poor Patnaik's values. Table 2: between Resnikoff's and ours. Included in Table 2 are also values of ν (center value) using Eq. (15) the cubic equation used by Resnikoff with Tippett's values of d_2 and d_3 for the coefficients. For some curious reasons, they check only approximately with Resnikoff's original results. Since we have an iterative program for Eq. (14), all we need to do here, is to substitute a subroutine of Eq. (15) in place of that for Eq. (14) and proceed with the computation. Therefore we must conclude that Resnikoff's values are in error. From these two tables, we see that both Patnaik's and Resnikoff's tables need some corrections, especially on values of ν for small n. Thomson's 9 pairs shown in Table 1 are superior to Patnaik's. They contain only occasional errors at the 4th decimal.

χ^2-Approximation Cox [3] approximated $\overline{W} = \overline{R}/\sigma'$ with $c\chi^2/\nu = c'\chi^2$ also by equating the first two moments of the latter with the former random variables and tabulated the values of $2c'(=\theta^2 d$, Cox's notation) and ν for m = 1 (only) and n = 2(1) 10. Since $E\chi^2 = \nu$ and Var $\chi^2 = 2\nu$, values of c and ν are trivial to solve. Equating the two first moments, we have,

$$c(\nu)/\nu = c = d_2 \tag{16}$$

Equating the two second moments, we have,

Table 1. Values of ν and c for Patnaik's χ-Approximation to $\overline{\overline{W}}$.

	m=1		m=2		m=3		m=4		m=5	
n	ν	c	ν	c	ν	c	ν	c	ν	c
2	1.00000 [1.0000]	1.414214 [1.41421]	1.9195+	1.27930	2.8173	1.23105+	3.7062	1.20620	4.5906	1.19105-
3	1.98463 (1.934) [1.9846]	1.911540 (1.9164) [1.9115-]	3.8337 (3.850)	1.80538 (1.8049)	5.6628 (5.674)	1.76857 (1.7684)	7.4854 (7.499)	1.74988 (1.7498)	9.3050+ (9.312)	1.73857 (1.7385)
4	2.92916 (2.951) [2.9291]	2.238865 (2.2374) [2.23887]	5.6935+ (5.705)	2.15069 (2.1505)	8.4415- (8.450)	2.12049 (2.1204)	11.1846 (11.191)	2.10522 (2.1052)	13.9356 (13.931)	2.09601 (2.0960)
5	3.82651 (3.828) [3.8267]	2.481246 (2.4812) [2.48124]	7.4710 (7.474)	2.40484 (2.4048)	11.1019 (11.107)	2.37883 (2.3788)	14.7288 (14.732)	2.36571 (2.3657)	18.3542 (18.355)	2.35781 (2.3578)
6	4.67716 (4.692) [4.6772]	2.672531 (2.6721) [2.67253]	9.1612 (9.179)	2.60439 (2.6001)	13.6335+ (13.641)	2.58127 (2.5812)	18.1026 (18.109)	2.56964 (2.5696)	22.5704 (22.576)	2.56263 (2.5626)
7	5.48415+ (5.499) [5.4841]	2.829802 (2.8295) [2.82981]	10.7675- (10.779)	2.76779 (2.7677)	16.0405- (16.052)	2.74681 (2.7468)	21.3107 (21.324)	2.73626 (2.7362)	26.5698 (26.596)	2.72991 (2.7299)
8	6.251123 (6.259) [6.2512]	2.962883 (2.9630) [2.96288]	12.2959 (12.297)	2.90562 (2.9056)	18.3315- (18.328)	2.88628 (2.8850)	24.3645+ (24.358)	2.87656 (2.8768)	30.3966 (30.386)	2.87071 (2.8707)
9	6.98207 (6.989) [6.9818]	3.077930 (3.0778) [3.07794]	13.7530 (13.753)	3.02446 (3.0245)	20.5162 (20.511)	3.00642 (3.0064)	27.2769 (27.270)	2.99737 (2.9974)	34.0367 (34.025)	2.99192 (2.9920)
10	7.68007 (7.689) [7.6798]	3.1790454+ (3.1789) [3.17905+]	15.1459 (15.151)	3.12869 (3.1287)	22.6041 (22.609)	3.11172 (3.1117)	30.0602 (30.066)	3.10320 (3.1032)	37.5156 (37.523)	3.09808 (3.0981)

Note: Values in parenthesis are Patnaik's [6], in brackets are Thomson's [13].
Values above them are ours.

Table 2. Values of ν and c for Resnikoff's χ-Approximation to $\overline{W}$.

mn	m	n	c	ν
3	1	3	1.9115+, 1.9117, (1.910)	1.9846, 1.9858, (1.934)
4	1	4	2.2389, 2.2389, (2.234)	2.922, 2.9315+, (2.995)
5	1	5	2.4812, 2.4813. (2.474)	3.8265+ 3.8284, (3.828)
7	1	7	2.8298, 2.8298, (2.830)	5.4842, 5.4856, (5.499)
10	2	5	2.4048, 2.4048, (2.405)	7.4710, 7.4717, (7.474)
15	3	5	2.3788, 2.3788, (2.379)	11,1019, 11.1019, (11,106)
25	5	5	2.3578, 2.3578, (2.358)	18.3542, 18.3536, (18.355)
30	6	5	2.3525+, 2.3535+, (2.353)	21.9787, 21.9780, (21.986)
35	7	5	2.3487, 2.3487, (2.349)	25.6026, 25.6019, (25.611)
40	8	5	2.3459, 2.3459 (2.346)	29.2264, 29.2255-, (29.236)
50	10	5	2.3419, 2.3419, (2.342)	36.4733, 36.4722, (36.486)
60	12	5	2.3393, 2.3393, (2.339)	43.7199, 43.7184, (43.735)
85	17	5	2.3354, 2.3354, (2.335)	61.8354, 61.833, (61.856)
115	23	5	2.3329, 2.3329, (2.333)	83.5733, 83.5704, (83.601)
175	35	5	2.3305+, 2.3305+, (2.331)	127.0482, 127.0439, (127.091)
230	46	5	2.3294, 2.3294, (2.330)	166.8999, 166.8942, (166.958)

Note: Center column from Eq. (15); values in parenthesis are Resnikoff's [7]. Values at left are ours rounded to 4 decimals.

$$(c/\nu)^2 (2_\nu) = 2c^2/\nu = 2d_2^2/\nu = d_3^2/m$$

or

$$\nu = 2m\,(d_2/d_3)^2 \qquad (17)$$

And hence,

$$c' = c/\nu = d_3^2/2md_2 \qquad (18)$$

Eqs. (16) and (17) represent a very small number of additional program steps and hence are easily incorporated into our computer program.

In a series of articles on the cumulative sum control charts by Johnson and Leone [8], they needed an approximation for the sample range and included a table of c' and ν for both Patnaik's χ-approximation as well as Cox's χ^2-approximation. Table 3 gives a comparison between values by Johnson and Leone and the results of our program using our more accurate d_2 and d_3 (Appendix A). By comparison Johnson and Leone's values are uneven and some with large differences in the second decimal place and therefore are obvious in need of corrections. The most recent entry for Patnaik's c and ν for m=1(1)15, 20, 30, 50 and n=2(1)15 can be found from Nelson [14]. Unfortunately, Nelson only tabulated them (just like David's and Duncan's) in 1 to 2 decimals. In Appendix B, we give our values for Patnaik's c and ν and in Appendix C, we give our values for Cox's ?c' and ν.

<u>Approximate Probability Integrals of W</u>

Since the density function as per Eq. (19) of the sample range from a normal population cannot be expressed in a closed form (except for n = 2, which is shown below), the probability integral, $P\ \{ W \leqq W \}$ can only be evaluated approximately. A table for this probability integral was

Table 3. Values of ν and c' for χ and χ^2- approximation to $\overline{W}$.
(Subscript 1 for Patnaik's χ-approximation and 2 for Cox's χ^2- approximation)

	χ-approximation (Patnaik's)		χ^2-approximation (Cox's)	
n	$c_1' = c_1/\sqrt{\nu}$	ν_1	$c_2' = c_2/\nu_2$	ν_2
2	1.4142	1.000	.3220	3.504
3	1.3569 (1.378)	1.985- (1.93)	.2331 (.233)	7.260 (7.27)
4	1.3081 (1.302)	2.929 (2.95)	.1880 (.188)	10.951 (10.95)
5	1.2684 (1.268)	3.827 (3.83)	.1605+ (.160)	14.491 (14.49)
6	1.2358 (1.237)	4.677 (4.69)	.1419 (.142)	17.863 (17.86)
7	1.2084 (1.207)	5.484 (5.50)	.1284 (.128)	21.069 (21.08)
8	1.1850+ (1.184)	6.251 (6.26)	.1180 (.118)	24.122 (24.11)
9	1.1648 (1.164)	6.982 (6.99)	.1099 (.110)	27.034 (27.01)
10	1.1471 (1.146)	7.680 (7.69)	.1032 (.103)	29.816 (29.82)

Note: Values in parenthesis are Johnson and Leone's [8].
Values above them are ours.

calculated by Pearson and Hartley [9] and reproduced in [4] as Table 23 for n = 2(1) 20 and W = 0(0.5) 7.25 in 4D. The density function of W is,

$$g(w) = n(n-1) \int_{-\infty}^{+\infty} [F(w+y) - F(y)]^{n-2} f(w+y)\, f(y)dy \qquad (19)$$

where, under the normal case, $F(y)$ is defined in Eq. (4), where $n = 2$, Eq. (19) may be integrated into:

$$g(w) = \frac{1}{\sqrt{\pi}} e^{-\frac{w^2}{4}}, \qquad \text{for } w > 0 \qquad (20)$$

which is a folded or half-normal density. Furthermore,

$$P\{W \leqq a\} = \int_0^a g(w)\, dw = \int^{\frac{a^2}{2}} y^{\frac{1}{2}-1} e^{-y/2} dy/2^{\frac{1}{2}}\Gamma(\tfrac{1}{2}) \qquad (21)$$

which is a χ^2-integral with one degree of freedom. In fact, the k^{th} moment in this case is,

$$EW^k = \frac{1}{\sqrt{\pi}} \int_0^\infty w^k e^{-\frac{w^2}{4}} dw = 2^k \Gamma\left(\frac{k+1}{2}\right) / \sqrt{\pi} \qquad (22)$$

from which one may find, for $n = 2$,

$$d_2 = EW = 2/\sqrt{\pi} \quad \text{and} \quad (d_3)^2 = \text{Var } W = 2 - 4/\pi . \qquad (23)$$

Eq. (23) is interesting because it shows as expected that for $n = 2$, $d_2 = 2c_2$ and $d_3 = 2c_3$ where $c_2 = \sqrt{2}\,\Gamma(n/2)/\sqrt{n}\,\Gamma(n/2-1/2)$ and $c_3 = \sqrt{(n-1)/n - c_2^2}$ are factor for sigma control charts widely used in quality control.

The following are probability integrals using the two approximations revisited by this report:

(a) Using Patnaik's χ-approximation, $W = c_1 \chi / \sqrt{\nu_1} = c_1' \chi$, the distribution of W may be found from $f(\chi)$ given by Eq. (5) as, (we omit for clarity the subscripts for c and ν)

$$f(w) = \frac{w^{\nu-1} e^{-w^2/2c'^2}}{c'^{\nu} 2^{\nu/2-1}\Gamma(\nu/2)}, \quad w > 0,\ \nu > 0. \tag{24}$$

From this, $P\{W \leqq a\} = \int_0^a f(w)dw$ can be obtained by letting $y = w^2/2c'^2$ as a gamma integral,

$$P\{W \leqq a\} = \int_0^{\frac{a^2}{a}c'^2} y^{\nu/2-1}\, e^{-y}\, dy/\Gamma(\nu/2) = G\left(\frac{a^2}{2c_1'^2} \,\middle|\, \frac{\nu}{2}\right) \tag{25}$$

Incidentally, by changing the variable of integration in Eq. (21) from y to $z = y/2$, the probability integral of w for $n = 2$ can be seen as $G\left(\frac{a^2}{4} \,\middle|\, \frac{1}{2}\right)$ which says that,

$$\nu_1 = 1,\ c_1 = c_1' = \sqrt{2} \quad \text{for } n = 2\,. \tag{26}$$

Of course Eq. (25) checks exactly the entries shown in Table 1.

(b) Using Cox's χ^2-approximation, $W = c_2\chi^2/\nu_2 = c_2'\chi^2$, the distribution of W may be found from a χ^2-distribution as,

$$f(w) = \frac{w^{\nu/2-1} e^{-w/2c'}}{(2c')^{\nu/2}\,\Gamma(\nu/2)}, \quad w > 0,\ \nu > 0. \tag{27}$$

By letting $y = w/2c'$, the probability integral for W is,

$$P\{W \leqq b\} = \int_0^{b/2c'} y^{\nu/2-1}\, e^{-y}/\Gamma(\nu/2) = G\left(\frac{b}{2c'} \,\middle|\, \frac{\nu}{2}\right) \tag{28}$$

With a gamma integral subroutine, both approximate probability integrals given by Eqs. (25) and (28) may be easily evaluated by inputting our more accurate values for c_1, ν_1, c_2 and ν_2 shown in Appendices B and C.

Pearson [10] made a comparison of the above two approximate probability integrals with those obtained by Table 23 of [4] which be called "True P.I.". Table 4 of this report reproduces his results together with the output of our program. The agreements are good and very few corrections are needed on Pearson's calculations. Had we used David's [11] or Johnson and Leone's [8] c and ν for inputs, the results will be way off.

Table 4. Probability Integral Approximation

n=4

W= R/σ'	True P. I.	χ-approx.	χ^2-approx.
.35	(.0053)	.00575+, (.0058)	.00112, (.0012)
.45	(.0111)	.01184, (.0119)	.00358, (.0036)
.75	(.0483)	.04972, (.0499)	.03063, (.0306)
1.00	(.1057)	.10724, (.1074)	.08716, (.0877)
1.30	(.2054)	.20631, (.2065)	.19727, (.1973)
2.00	(.5096)	.50786, (.5079)	.53026, (.5303)
2.80	(.8045)	.80318, (.8031)	.81532, (.8153)
3.25	(.9016)	.90144, (.9013)	.90207, (.9021)
3.65	(.9516)	.95207, (.9520)	.94702, (.9470)
4.40	(.9899)	.99051, (.9904)	.98487, (.9849)
4.70	(.9951)	.99549, (.9955)	.99111, (.9911)

n=6

W= R/σ'	True P. I.	χ-approx.	χ^2-approx.
.75	(.0050)	.00602, (.0060)	.00182, (.0018)
.90	(.0117)	.01336, (.0134)	.00584, (.0058)
1.25	(.0495)	.05252, (.0526)	.03813, (.0383)
1.50	(.1031)	.10600, (.1061)	.09250+, (.0925)
1.80	(.2000)	.20116, (.2012)	.19691, (.1969)
2.45	(.4899)	.48572, (.4858)	.50457, (.5046)
3.25	(.8053)	.80303, (.8030)	.81186, (.8119)
3.65	(.8981)	.89817, (.8982)	.89775+, (.8978)
4.05	(.9519)	.95332, (.9533)	.94809, (.9481)
4.75	(.9898)	.99104, (.9910)	.98616, (.9862)
5.05	(.9952)	.99605- (.9960)	.99249, (.9925)

n=10

W= R/σ'	True P. I.	χ-approx.	χ^2-approx.
1.35	(.0054)	.00750-, (.0076)	.00339, (.0034)
1.50	(.0117)	.01485, (.0149)	.00847, (.0085)
1.85	(.0479)	.05300, (.0530)	.04255-, (.0425)
2.10	(.1015)	.10604, (.1061)	.09708, (.0971)
2.40	(.2025)	.20338, (.2034)	.20222, (.2022)
3.00	(.4878)	.47994, (.4800)	.49544, (.4954)
3.75	(.8602)	.80268, (.8026)	.80896, (.8089)
4.15	(.9038)	.90476, (.9047)	.90305+, (.9030)
4.45	(.9474)	.94997, (.9500)	.94528, (.9453)
5.15	(.9898)	.99193, (.9919)	.98799, (.9880)
5.40	(.9948)	.99623, (.9962)	.99338, (.9934)

n=15

W= R/σ	True P. I.	χ-approx.	χ^2-approx.
1.80	(.0049)	.00773, (.0077)	.00411, (.0041)
1.95	(.0108)	.01506, (.0150)	.00963, (.0096)
2.30	(.0468)	.05355-, (.0535)	.04512, (0451)
2.55	(.1026)	.10825-, (.1083)	.10150+, (.1015)
2.85	(.2103)	.21051, (.2105)	.21087, (.2109)
3.40	(.4885)	.47751, (.4775)	.49089, (.4909)
4.10	(.8036)	.79906, (.7990)	.80426, (.8043)
4.45	(.8964)	.89753, (.8975)	.89614, (.8962)
4.80	(.9505)	.95424, (.9543)	.94977, (.9498)
5.45	(.9900)	.99279, (.9928)	.98950+, (.9895)
5.70	(.9950)	.99686, (.9968)	.99462, (.9946)

Note: Values in the parenthesis are Pearson's [10]. Values in front of them are ours.

REFERENCES

1. Grubbs, F. E. and Weaver, C. L. "The Best Unbaised Estimate of Population Standard Deviation based on group ranges, "Journal of American Statistical Association vol. 42, (1947), pp. 224-241.

2. Cox, D. R., "The use of the Range in Sequential Analysis", Journal of the Royal Statistical Society, Series B, Vol. 11 (1949), pp. 101-114.

3. Tippett, L. H. C., "On the Extreme Individuals and the Range of Samples Taken From a Normal Population", Biometrika, Vol. 17 (1925), pp. 364-387.

4. Pearson, E. S. and Hartley, H. O. (Editors) Biometrika Tables for Statisticians, University Press, Cambridge (England), 1956, (1st Edition, 1954).

5. Kao, J. H. K. "Microcomputer and QC Statistics: Programs and Tables", ONR Technical Report, Department of Industrial Engineering and Operations Research, New York University, 1971. Also Proceedings of the 1973 Annual Reliability and Maintainability symposium.

6. Patnaik, P. B., "The Use of Mean Range as an Estimator of Variance in Statistical Tests", Biometrika, Vol. 37 (1950) pp. 78-87.

7. Resnikoff, G. J., "The distribution of the Average Range for Sub-groups of Five", Technical Report No. 15, Stanford University, 1954, with excerpts in Mathematical and Statistical Principles Underlying Military Standard 414, Office of the Assistant Secretary of Defense, Washington, D. C. 1958. pp. 23-24.

8. Johnson, N. L. and Leone, F. C. "Cumulative Sum Chart, II", Industrial Quality Control, Vol. 19, No. 1 (July, 1962) pp. 29-36. (Others of the same series: I in vol. 18, no. 12 (June, 1962) pp. 15-21 and III in Vol. 19, no. 2 (Aug., 1962), pp. 22-28.

9. Pearson, E. S. and Hartley, H. O., "The Probability Integral of the Range in Samples of n observations from the Normal Population", Biometrika, Vol. 32 (1942) pp. 301-310, P $(W \leq w)$ for $n = 2(1)20$, $w = 0(.05)7.25$ in 5D (Also as Table 23 of Ref. 4 above).

10. Pearson, E. S., "Comparison of two Approximation to the Distribution of the Range in Small Samples from Normal Populations", Biometrika, Vol. 39 (1952), pp. 130-136.

11. David, H. A., "Further Applications of Range to the Analysis of Variance", Biometrika Vol. 38 (1951), pp. 393-409.

12. Duncan, A. J., "Design and Operation of a Double-limit Variables Sampling Plan", Journal of American Statistical Association, Vol. 53 (1958), pp. 543-50.

13. Thomson, G. W., "Scale Factor and Degrees of Freedom for Small Sample Sizes for χ-approximation to the Range", Biometrika, Vol. 40 (1953), pp. 449-50.

14. Nelson, L. S., "Use of the Range to Estimate Variability", Journal of Quality Technology, Vol. 7 (1975), pp. 46-48.

15. TEICHROEW, D., "Tables of Expected Values of Order Statistics and Products of order statistics for samples of size 20 and less from the Normal distribution", Annals of Mathematical Statistics, Vol. 27 (1956), pp. 410-26.

16. GASCOIGNE, J. C. and Hill, I. D., "The Draft British Standard BS6002: Sampling Plans by Variables", Journal of Royal Statistical Society (Section A, Part 3) Vol. 139 (1976), pp. 299-317.

Appendix A. Mean d_2 and Standard Deviation d_3 of the Standard Range W from a Normal Population

n	EW = d_2	Std W = d_3	n	EW = d_2	Std W = d_3
-	-	-	26	3.9643157	.70498834
2	1.1283792	.85250247	27	3.9965386	.70169659
3	1.6925688	.88836800	28	4.0274138	.69855282
4	2.0587507	.87980820	29	4.0570443	.69554570
5	2.3259289	.86408194	30	4.0855217	.69266510
6	2.5344127	.84803967	31	4.1129282	.68990192
7	2.7043568	.83320534	32	4.1393377	.68724797
8	2.8472006	.81983149	33	4.1648167	.68469583
9	2.9700263	.80783427	34	4.1894255+	.68223881
10	3.0775055-	.79705067	35	4.2132189	.67987079
11	3.1728727	.78731462	36	4.2362466	.67758623
12	3.2584553	.77847834	37	4.2585541	.67538004
13	3.3359804	.77041620	38	4.2801829	.67324760
14	3.4067631	.76302310	39	4.3011713	.67118462
15	3.4718269	.75621143	40	4.3215544	.66918720
16	3.5319828	.74990809	41	4.3413644	.66725172
17	3.5878840	.74405178	42	4.3606312	.66537485-
18	3.6400638	.73859085+	43	4.3793825+	.66355350+
19	3.6889630	.73348150-	44	4.3976439	.66178482
20	3.7349501	.72868635-	45	4.4154391	.66006617
21	3.7783358	.72417334	46	4.4327903	.65839507
22	3.8193846	.71991481	47	4.4497181	.65676923
23	3.8583234	.71588674	48	4.4662418	.65518651
24	3.8953481	.71206818	49	4.4823792	.65364492
25	3.9306292	.70844077	50	4.4981473	.65214259

APPENDIX B PATNAIK'S CHI APPROXIMATION

N	DF(M=1)	C(M=1)	DF(M=2)	C(M=2)	DF(M=3)	C(M=3)
2	1.0000000D 00	1.4142136D 00	1.9195218D 00	1.2793044D 00	2.8172896D 00	1.2310537D 00
3	1.9846345D 00	1.9115404D 00	3.8337216D 00	1.8053775D 00	5.6627792D 00	1.7685743D 00
4	2.9291550D 00	2.2388651D 00	5.6935391D 00	2.1506942D 00	8.4414647D 00	2.1204894D 00
5	3.8265132D 00	2.4812462D 00	7.4710477D 00	2.4048418D 00	1.1101853D 01	2.3788284D 00
6	4.6771607D 00	2.6725305D 00	9.1612114D 00	2.6043873D 00	1.3633502D 01	2.5812732D 00
7	5.4841533D 00	2.8298016D 00	1.0767465D 01	2.7677900D 00	1.6040465D 01	2.7468083D 00
8	6.2512253D 00	2.9628829D 00	1.2295944D 01	2.9056175D 00	1.8331451D 01	2.8862766D 00
9	6.9820658D 00	3.0779299D 00	1.3753301D 01	3.0244593D 00	2.0516182D 01	3.0064245D 00
10	7.6800658D 00	3.1790454D 00	1.5145891D 01	3.1286874D 00	2.2604051D 01	3.1117203D 00
11	8.3482531D 00	3.2690955D 00	1.6479508D 01	3.2213434D 00	2.4603661D 01	3.2052679D 00
12	8.9893018D 00	3.3501581D 00	1.7759327D 01	3.3046248D 00	2.6522720D 01	3.2893070D 00
13	9.6055619D 00	3.4237854D 00	1.8989931D 01	3.3801680D 00	2.8368068D 01	3.3655033D 00
14	1.0199099D 01	3.4911659D 00	2.0175372D 01	3.4492227D 00	3.0145753D 01	3.4351278D 00
15	1.0771736D 01	3.5532292D 00	2.1319235D 01	3.5127639D 00	3.1861139D 01	3.4991714D 00
16	1.1325083D 01	3.6107153D 00	2.2424695D 01	3.5715660D 00	3.3518972D 01	3.5584205D 00
17	1.1860566D 01	3.6642222D 00	2.3494576D 01	3.6262540D 00	3.5123481D 01	3.6135093D 00
18	1.2379459D 01	3.7142403D 00	2.4531397D 01	3.6773391D 00	3.6678434D 01	3.6649561D 00
19	1.2882898D 01	3.7611758D 00	2.5537411D 01	3.7252444D 00	3.8187207D 01	3.7131900D 00
20	1.3371905D 01	3.8053694D 00	2.6514648D 01	3.7703241D 00	3.9652839D 01	3.7585698D 00
21	1.3847399D 01	3.8471089D 00	2.7464931D 01	3.8128773D 00	4.1078062D 01	3.8013983D 00
22	1.4310213D 01	3.8866407D 00	2.8389915D 01	3.8531594D 00	4.2465355D 01	3.8419341D 00
23	1.4761096D 01	3.9241752D 00	2.9291094D 01	3.8913884D 00	4.3816955D 01	3.8803980D 00
24	1.5200737D 01	3.9598957D 00	3.0169837D 01	3.9277542D 00	4.5134915D 01	3.9169818D 00
25	1.5629765D 01	3.9939618D 00	3.1027394D 01	3.9624218D 00	4.6421109D 01	3.9518526D 00
26	1.6048749D 01	4.0265128D 00	3.1864901D 01	3.9955351D 00	4.7677239D 01	3.9851557D 00

APPENDIX B PATNAIK'S CHI APPROXIMATION

N	DF(M=1)	C(M=1)	DF(M=2)	C(M=2)	DF(M=3)	C(M=3)
26	1.60487490 01	4.0265128D 00	3.1864901D 01	3.9955351D 00	4.7677239D 01	3.9851557D 00
27	1.6458212D 01	4.0576712D 00	3.2683399D 01	4.0272207D 00	4.8904864D 01	4.0170192D 00
28	1.6858644D 01	4.0875463D 00	3.3483865D 01	4.0575912D 00	5.0105449D 01	4.0475569D 00
29	1.7250490D 01	4.1162349D 00	3.4267182D 01	4.0867466D 00	5.1280320D 01	4.0768698D 00
30	1.7634162D 01	4.1438234D 00	3.5034178D 01	4.1147758D 00	5.2430713D 01	4.1050476D 00
31	1.8010028D 01	4.1703881D 00	3.5785583D 01	4.1417574D 00	5.3557728D 01	4.1321698D 00
32	1.8378469D 01	4.1960011D 00	3.6522157D 01	4.1677656D 00	5.4662501D 01	4.1583112D 00
33	1.8739791D 01	4.2207227D 00	3.7244508D 01	4.1928625D 00	5.5745946D 01	4.1835345D 00
34	1.9094321D 01	4.2446125D 00	3.7953294D 01	4.2171093D 00	5.6809047D 01	4.2079016D 00
35	1.9442326D 01	4.2677202D 00	3.8649044D 01	4.2405570D 00	5.7852599D 01	4.2314639D 00
36	1.9784087D 01	4.2900937D 00	3.9332319D 01	4.2632549D 00	5.8877441D 01	4.2542710D 00
37	2.0119854D 01	4.3117764D 00	4.0003618D 01	4.2852476D 00	5.9884324D 01	4.2763681D 00
38	2.0449862D 01	4.3328083D 00	4.0663413D 01	4.3065759D 00	6.0873952D 01	4.2977962D 00
39	2.0774321D 01	4.3532235D 00	4.1312118D 01	4.3272752D 00	6.1846950D 01	4.3185911D 00
40	2.1093465D 01	4.3730587D 00	4.1950203D 01	4.3473827D 00	6.2804020D 01	4.3387903D 00
41	2.1407464D 01	4.3923417D 00	4.2578009D 01	4.3669272D 00	6.3745675D 01	4.3584227D 00
42	2.1716521D 01	4.4111026D 00	4.3195937D 01	4.3859394D 00	6.4672515D 01	4.3775195D 00
43	2.2020797D 01	4.4293669D 00	4.3804313D 01	4.4044454D 00	6.5585029D 01	4.3961069D 00
44	2.2320464D 01	4.4471591D 00	4.4403477D 01	4.4224704D 00	6.6483727D 01	4.4142101D 00
45	2.2615683D 01	4.4645029D 00	4.4993751D 01	4.4400385D 00	6.7369092D 01	4.4318537D 00
46	2.2906591D 01	4.4814182D 00	4.5575412D 01	4.4571703D 00	6.8241539D 01	4.4490583D 00
47	2.3193337D 01	4.4979253D 00	4.6148752D 01	4.4738863D 00	6.9101508D 01	4.4658446D 00
48	2.3476051D 01	4.5140422D 00	4.6714037D 01	4.4902051D 00	6.9949394D 01	4.4822313D 00
49	2.3754867D 01	4.5297873D 00	4.7271529D 01	4.5061455D 00	7.0785592D 01	4.4982373D 00
50	2.4029895D 01	4.5451751D 00	4.7821450D 01	4.5217222D 00	7.1610436D 01	4.5138775D 00

APPENDIX B PATNAIK'S CHI APPROXIMATION

N	DF(M=4)	C(M=4)	DF(M=5)	C(M=5)	DF(M=6)	C(M=6)
2	3.7061654D 00	1.2062047D 00	4.5906034D 00	1.1910465D 00	5.4725311D 00	1.1808329D 00
3	7.4853527D 00	1.7498824D 00	9.3050652D 00	1.7385709D 00	1.1123274D 01	1.7309888D 00
4	1.1184553D 01	2.1052245D 00	1.3925593D 01	2.0960122D 00	1.6665579D 01	2.0898480D 00
5	1.4728813D 01	2.3657144D 00	1.8354174D 01	2.3578110D 00	2.1978719D 01	2.3525273D 00
6	1.8102587D 01	2.5696382D 00	2.2570352D 01	2.5626318D 00	2.7037447D 01	2.5579503D 00
7	2.1310703D 01	2.7362572D 00	2.6579809D 01	2.7299069D 00	3.1848343D 01	2.7256652D 00
8	2.4364518D 01	2.8765573D 00	3.0396589D 01	2.8707100D 00	3.6428158D 01	2.8668052D 00
9	2.7276869D 01	2.9973664D 00	3.4036663D 01	2.9919184D 00	4.0796008D 01	2.9882808D 00
10	3.0060210D 01	3.1032020D 00	3.7515557D 01	3.0980797D 00	4.4970495D 01	3.0946602D 00
11	3.2725969D 01	3.1971999D 00	4.0847529D 01	3.1923493D 00	4.8968712D 01	3.1891115D 00
12	3.5284397D 01	3.2816213D 00	4.4045379D 01	3.2770012D 00	5.2806011D 01	3.2739175D 00
13	3.7744595D 01	3.3581469D 00	4.7120472D 01	3.3537253D 00	5.6496021D 01	3.3507744D 00
14	4.0114618D 01	3.4280586D 00	5.0082869D 01	3.4238101D 00	6.0050812D 01	3.4209748D 00
15	4.2401605D 01	3.4923554D 00	5.2941490D 01	3.4882593D 00	6.3481084D 01	3.4855260D 00
16	4.4611881D 01	3.5518295D 00	5.5704238D 01	3.5478691D 00	6.6796317D 01	3.5452263D 00
17	4.6751078D 01	3.6071200D 00	5.8378148D 01	3.6032810D 00	7.0004953D 01	3.6007194D 00
18	4.8824218D 01	3.6587489D 00	6.0969497D 01	3.6550195D 00	7.3114522D 01	3.6525312D 00
19	5.0835799D 01	3.7071481D 00	6.3483904D 01	3.7035182D 00	7.6131766D 01	3.7010963D 00
20	5.2789869D 01	3.7526788D 00	6.5926431D 01	3.7491398D 00	7.9062758D 01	3.7467786D 00
21	5.4690071D 01	3.7956457D 00	6.8301628D 01	3.7921900D 00	8.1912959D 01	3.7898845D 00
22	5.6539709D 01	3.8363091D 00	7.0613625D 01	3.8329302D 00	8.4687322D 01	3.8306759D 00
23	5.8341763D 01	3.8748910D 00	7.2866148D 01	3.8715831D 00	8.7390319D 01	3.8693763D 00
24	6.0098971D 01	3.9115845D 00	7.5062615D 01	3.9083425D 00	9.0026053D 01	3.9061797D 00
25	6.1813830D 01	3.9465573D 00	7.7206150D 01	3.9433768D 00	9.2598269D 01	3.9412550D 00
26	6.3488608D 01	3.9799559D 00	7.9299587D 01	3.9768327D 00	9.5110370D 01	3.9747492D 00

APPENDIX B PATNAIK'S CHI APPROXIMATION

N	DF(M=4)	C(M=4)	DF(M=5)	C(M=5)	DF(M=6)	C(M=6)
26	6.3488608D 01	3.9799559D 00	7.9299587D 01	3.9768327D 00	9.5110370D 01	3.9747492D 00
27	6.5125384D 01	4.0119087D 00	8.1345524D 01	4.0088393D 00	9.7565472D 01	4.0067918D 00
28	6.6726110D 01	4.0425304D 00	8.3346400D 01	4.0395116D 00	9.9966503D 01	4.0374977D 00
29	6.8292555D 01	4.0719224D 00	8.5304427D 01	4.0689511D 00	1.0231612D 02	4.0669690D 00
30	6.9826366D 01	4.1001748D 00	8.7221664D 01	4.0972484D 00	1.0461678D 02	4.0952963D 00
31	7.1329009D 01	4.1273676D 00	8.9099943D 01	4.1244836D 00	1.0687070D 02	4.1225598D 00
32	7.2801999D 01	4.1535759D 00	9.0941156D 01	4.1507321D 00	1.0908014D 02	4.1488352D 00
33	7.4246553D 01	4.1788627D 00	9.2746827D 01	4.1760571D 00	1.1124693D 02	4.1741857D 00
34	7.5663985D 01	4.2032902D 00	9.4518595D 01	4.2005209D 00	1.1337304D 02	4.1986737D 00
35	7.7055352D 01	4.2269100D 00	9.6257784D 01	4.2241753D 00	1.1546005D 02	4.2223511D 00
36	7.8421776D 01	4.2497720D 00	9.7965794D 01	4.2470702D 00	1.1750965D 02	4.2452681D 00
37	7.9764255D 01	4.2719214D 00	9.9643875D 01	4.2692512D 00	1.1952334D 02	4.2674701D 00
38	8.1083730D 01	4.2933997D 00	1.0129320D 02	4.2907596D 00	1.2150252D 02	4.2889986D 00
39	8.2381033D 01	4.3142425D 00	1.0291481D 02	4.3116313D 00	1.2344844D 02	4.3098895D 00
40	8.3657099D 01	4.3344877D 00	1.0450988D 02	4.3319041D 00	1.2536251D 02	4.3301808D 00
41	8.4912613D 01	4.3541643D 00	1.0607926D 02	4.3516072D 00	1.2724576D 02	4.3499017D 00
42	8.6148375D 01	4.3733035D 00	1.0762395D 02	4.3707719D 00	1.2909937D 02	4.3690834D 00
43	8.7365037D 01	4.3919317D 00	1.0914476D 02	4.3894246D 00	1.3092434D 02	4.3877525D 00
44	8.8563279D 01	4.4100742D 00	1.1064255D 02	4.4075908D 00	1.3272168D 02	4.4059344D 00
45	8.9743743D 01	4.4277557D 00	1.1211812D 02	4.4252950D 00	1.3449235D 02	4.4236538D 00
46	9.0906986D 01	4.4449968D 00	1.1357216D 02	4.4425581D 00	1.3623720D 02	4.4409315D 00
47	9.2053591D 01	4.4618183D 00	1.1500540D 02	4.4594008D 00	1.3795708D 02	4.4577884D 00
48	9.3184086D 01	4.4782390D 00	1.1641851D 02	4.4758420D 00	1.3965280D 02	4.4742432D 00
49	9.4298999D 01	4.4942779D 00	1.1781214D 02	4.4919007D 00	1.4132515D 02	4.4903151D 00
50	9.5398774D 01	4.5099500D 00	1.1918685D 02	4.5075919D 00	1.4297480D 02	4.5060191D 00

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=1)	2C'(M=1)	DF(M=2)	2C'(M=2)	DF(M=3)	2C'(M=3)
2	3.5038770D 00	6.4407467D-01	7.0077539D 00	3.2203734D-01	1.0511631D 01	2.1469156D-01
3	7.2600037D 00	4.6627216D-01	1.4520007D 01	2.3313608D-01	2.1780011D 01	1.5542405D-01
4	1.0951195D 01	3.7598650D-01	2.1902390D 01	1.8799325D-01	3.2853584D 01	1.2532883D-01
5	1.4491489D 01	3.2100620D-01	2.8982978D 01	1.6050310D-01	4.3474467D 01	1.0700207D-01
6	1.7862915D 01	2.8376250D-01	3.5725830D 01	1.4188125D-01	5.3588745D 01	9.4587500D-02
7	2.1069483D 01	2.5670841D-01	4.2138967D 01	1.2835421D-01	6.3208450D 01	8.5569471D-02
8	2.4122201D 01	2.3606474D-01	4.8244403D 01	1.1803237D-01	7.2366604D 01	7.8688247D-02
9	2.7033734D 01	2.1972742D-01	5.4067468D 01	1.0986371D-01	8.1101203D 01	7.3242472D-02
10	2.9816441D 01	2.0643010D-01	5.9632883D 01	1.0321505D-01	8.9449324D 01	6.8810034D-02
11	3.2481693D 01	1.9536375D-01	6.4963386D 01	9.7681875D-02	9.7445079D 01	6.5121250D-02
12	3.5039707D 01	1.8598645D-01	7.0079414D 01	9.2993224D-02	1.0511912D 02	6.1995483D-02
13	3.7499559D 01	1.7792105D-01	7.4999119D 01	8.8960523D-02	1.1249868D 02	5.9307015D-02
14	3.9869269D 01	1.7089661D-01	7.9738578D 01	8.5448303D-02	1.1960787D 02	5.6965535D-02
15	4.2156025D 01	1.6471320D-01	8.4312049D 01	8.2356601D-02	1.2646807D 02	5.4904401D-02
16	4.4366082D 01	1.5921996D-01	8.8732164D 01	7.9609978D-02	1.3309825D 02	5.3073318D-02
17	4.6505087D 01	1.5430071D-01	9.3010174D 01	7.7150355D-02	1.3951526D 02	5.1433570D-02
18	4.8578057D 01	1.4986453D-01	9.7156114D 01	7.4932264D-02	1.4573417D 02	4.9954843D-02
19	5.0585486D 01	1.4583912D-01	1.0117897D 02	7.2919559D-02	1.5176846D 02	4.8613039D-02
20	5.2543420D 01	1.4216624D-01	1.0508684D 02	7.1083118D-02	1.5763026D 02	4.7388745D-02
21	5.4443498D 01	1.3879841D-01	1.0888700D 02	6.9399206D-02	1.6333050D 02	4.6266137D-02
22	5.6293024D 01	1.3569655D-01	1.1258605D 02	6.7848275D-02	1.6887907D 02	4.5232183D-02
23	5.8094976D 01	1.3282812D-01	1.1618995D 02	6.6414058D-02	1.7428493D 02	4.4276038D-02
24	5.9852090D 01	1.3016580D-01	1.1970418D 02	6.5082900D-02	1.7955627D 02	4.3388600D-02
25	6.1566862D 01	1.2768651D-01	1.2313372D 02	6.3843253D-02	1.8470059D 02	4.2562169D-02
26	6.3241561D 01	1.2537058D-01	1.2648312D 02	6.2685288D-02	1.8972468D 02	4.1790192D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=1)	2C'(M=1)	DF(M=2)	2C'(M=2)	DF(M=3)	2C'(M=3)
26	6.3241561D 01	1.2537058D-01	1.2648312D 02	6.2685288D-02	1.8972468D 02	4.1790192D-02
27	6.4878263D 01	1.2320115D-01	1.2975653D 02	6.1600573D-02	1.9463479D 02	4.1067049D-02
28	6.6478919D 01	1.2116362D-01	1.3295784D 02	6.0581812D-02	1.9943676D 02	4.0387874D-02
29	6.8045300D 01	1.1924539D-01	1.3609060D 02	5.9622693D-02	2.0413590D 02	3.9748462D-02
30	6.9579051D 01	1.1743540D-01	1.3915810D 02	5.8717698D-02	2.0873715D 02	3.9145132D-02
31	7.1081637D 01	1.1572405D-01	1.4216327D 02	5.7862024D-02	2.1324491D 02	3.8574683D-02
32	7.2554574D 01	1.1410273D-01	1.4510915D 02	5.7051366D-02	2.1766372D 02	3.8034244D-02
33	7.3999078D 01	1.1256399D-01	1.4799816D 02	5.6281997D-02	2.2199724D 02	3.7521331D-02
34	7.5416462D 01	1.1110109D-01	1.5083292D 02	5.5550543D-02	2.2624939D 02	3.7033696D-02
35	7.6807785D 01	1.0970812D-01	1.5361557D 02	5.4854058D-02	2.3042336D 02	3.6569372D-02
36	7.8174167D 01	1.0837969D-01	1.5634833D 02	5.4189847D-02	2.3452250D 02	3.6126565D-02
37	7.9516605D 01	1.0711105D-01	1.5903321D 02	5.3555524D-02	2.3854982D 02	3.5703683D-02
38	8.0836042D 01	1.0589788D-01	1.6167208D 02	5.2948941D-02	2.4250813D 02	3.5299294D-02
39	8.2133308D 01	1.0473632D-01	1.6426662D 02	5.2368161D-02	2.4639992D 02	3.4912108D-02
40	8.3409340D 01	1.0362279D-01	1.6681868D 02	5.1811394D-02	2.5022802D 02	3.4540929D-02
41	8.4664821D 01	1.0255414D-01	1.6932964D 02	5.1277070D-02	2.5399446D 02	3.4184713D-02
42	8.5900551D 01	1.0152743D-01	1.7180110D 02	5.0763714D-02	2.5770165D 02	3.3842476D-02
43	8.7117183D 01	1.0054003D-01	1.7423437D 02	5.0270016D-02	2.6135155D 02	3.3513344D-02
44	8.8315396D 01	9.9589501D-02	1.7663079D 02	4.9794750D-02	2.6494619D 02	3.3196500D-02
45	8.9495832D 01	9.8673615D-02	1.7899166D 02	4.9336807D-02	2.6848750D 02	3.2891205D-02
46	9.0659048D 01	9.7790345D-02	1.8131810D 02	4.8895173D-02	2.7197714D 02	3.2596782D-02
47	9.1805627D 01	9.6937795D-02	1.8361125D 02	4.8468897D-02	2.7541688D 02	3.2312598D-02
48	9.2936098D 01	9.6114233D-02	1.8587220D 02	4.8057116D-02	2.7880829D 02	3.2038078D-02
49	9.4050987D 01	9.5318063D-02	1.8810197D 02	4.7659032D-02	2.8215296D 02	3.1772688D-02
50	9.5150739D 01	9.4547810D-02	1.9030148D 02	4.7273905D-02	2.8545222D 02	3.1515937D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=4)	2C'(M=4)	DF(M=5)	2C'(M=5)	DF(M=6)	2C'(M=6)
2	1.4015508D 01	1.6101867D-01	1.7519385D 01	1.2881493D-01	2.1023262D 01	1.0734578D-01
3	2.9040015D 01	1.1656804D-01	3.6300019D 01	9.3254431D-02	4.3560022D 01	7.7712026D-02
4	4.3804779D 01	9.3996625D-02	5.4755974D 01	7.5197300D-02	6.5707169D 01	6.2664417D-02
5	5.7965956D 01	8.0251550D-02	7.2457445D 01	6.4201240D-02	8.6948934D 01	5.3501033D-02
6	7.1451660D 01	7.0940625D-02	8.9314575D 01	5.6752500D-02	1.0717749D 02	4.7293750D-02
7	8.4277933D 01	6.4177103D-02	1.0534742D 02	5.1341682D-02	1.2641690D 02	4.2784735D-02
8	9.6488805D 01	5.9016185D-02	1.2061101D 02	4.7212948D-02	1.4473321D 02	3.9344123D-02
9	1.0813494D 02	5.4931854D-02	1.3516867D 02	4.3945483D-02	1.6220241D 02	3.6621236D-02
10	1.1926577D 02	5.1607525D-02	1.4908221D 02	4.1286020D-02	1.7889865D 02	3.4405017D-02
11	1.2992677D 02	4.8840938D-02	1.6240847D 02	3.9072750D-02	1.9489016D 02	3.2560625D-02
12	1.4015883D 02	4.6496612D-02	1.7519853D 02	3.7197290D-02	2.1023824D 02	3.0997741D-02
13	1.4999824D 02	4.4480261D-02	1.8749780D 02	3.5584209D-02	2.2499736D 02	2.9653508D-02
14	1.5947716D 02	4.2724151D-02	1.9934645D 02	3.4179321D-02	2.3921573D 02	2.8482768D-02
15	1.6862410D 02	4.1178301D-02	2.1078012D 02	3.2942640D-02	2.5293615D 02	2.7452200D-02
16	1.7746433D 02	3.9804989D-02	2.2183041D 02	3.1843991D-02	2.6619649D 02	2.6536659D-02
17	1.8602035D 02	3.8575178D-02	2.3252544D 02	3.0860142D-02	2.7903052D 02	2.5716785D-02
18	1.9431223D 02	3.7466132D-02	2.4289029D 02	2.9972906D-02	2.9146834D 02	2.4977421D-02
19	2.0235794D 02	3.6459780D-02	2.5294743D 02	2.9167824D-02	3.0353691D 02	2.4306520D-02
20	2.1017368D 02	3.5541559D-02	2.6271710D 02	2.8433247D-02	3.1526052D 02	2.3694373D-02
21	2.1777399D 02	3.4699603D-02	2.7221749D 02	2.7759682D-02	3.2666099D 02	2.3133069D-02
22	2.2517210D C2	3.3924138D-02	2.8146512D 02	2.7139310D-02	3.3775814D 02	2.2616092D-02
23	2.3237991D 02	3.3207029D-02	2.9047488D 02	2.6565623D-02	3.4856986D 02	2.2138019D-02
24	2.3940836D 02	3.2541450D-02	2.9926045D 02	2.6033160D-02	3.5911254D 02	2.1694300D-02
25	2.4626745D 02	3.1921627D-02	3.0783431D 02	2.5537301D-02	3.6940117D 02	2.1281084D-02
26	2.5296624D 02	3.1342644D-02	3.1620780D 02	2.5074115D-02	3.7944936D 02	2.0895096D-02

APPENDIX C COX'S CHI-SQUARE APPROXIMATION

N	DF(M=4)	2C'(M=4)	DF(M=5)	2C'(M=5)	DF(M=6)	2C'(M=6)
26	2.5296624D 02	3.1342644D-02	3.1620780D 02	2.5074115D-02	3.7944936D 02	2.0895096D-02
27	2.5951305D 02	3.0800287D-02	3.2439131D 02	2.4640229D-02	3.8926958D 02	2.0533524D-02
28	2.6591568D 02	3.0290906D-02	3.3239460D 02	2.4232725D-02	3.9887352D 02	2.0193937D-02
29	2.7218120D 02	2.9811347D-02	3.4022650D 02	2.3849077D-02	4.0827180D 02	1.9874231D-02
30	2.7831620D 02	2.9358849D-02	3.4789525D 02	2.3487079D-02	4.1747431D 02	1.9572566D-02
31	2.8432655D 02	2.8931012D-02	3.5540819D 02	2.3144810D-02	4.2648982D 02	1.9287341D-02
32	2.9021830D 02	2.8525683D-02	3.6277287D 02	2.2820546D-02	4.3532744D 02	1.9017122D-02
33	2.9599631D C2	2.8140998D-02	3.6999539D 02	2.2512799D-02	4.4399447D 02	1.8760666D-02
34	3.0166585D 02	2.7775272D-02	3.7708231D 02	2.2220217D-02	4.5249877D 02	1.8516848D-02
35	3.0723114D 02	2.7427029D-02	3.8403893D 02	2.1941623D-02	4.6084671D 02	1.8284686D-02
36	3.1269667D 02	2.7094923D-02	3.9087083D 02	2.1675939D-02	4.6904500D 02	1.8063282D-02
37	3.1806642D 02	2.6777762D-02	3.9758303D 02	2.1422210D-02	4.7709963D 02	1.7851841D-02
38	3.2334417D 02	2.6474471D-02	4.0418021D 02	2.1179576D-02	4.8501625D 02	1.7649647D-02
39	3.2853323D 02	2.6184081D-02	4.1066654D 02	2.0947265D-02	4.9279985D 02	1.7456054D-02
40	3.3363736D 02	2.5905697D-02	4.1704670D 02	2.0724558D-02	5.0045604D 02	1.7270465D-02
41	3.3865928D 02	2.5638535D-02	4.2332410D 02	2.0510828D-02	5.0798892D 02	1.7092357D-02
42	3.4360221D 02	2.5381857D-02	4.2950276D 02	2.0305485D-02	5.1540331D 02	1.6921238D-02
43	3.4846873D 02	2.5135008D-02	4.3558591D 02	2.0108006D-02	5.2270310D 02	1.6756672D-02
44	3.5326158D 02	2.4897375D-02	4.4157698D 02	1.9917900D-02	5.2989238D 02	1.6598250D-02
45	3.5798333D 02	2.4668404D-02	4.4747916D 02	1.9734723D-02	5.3697499D 02	1.6445602D-02
46	3.6263619D 02	2.4447586D-02	4.5329524D 02	1.9558069D-02	5.4395429D 02	1.6298391D-02
47	3.6722251D 02	2.4234449D-02	4.5902814D 02	1.9387559D-02	5.5083376D 02	1.6156299D-02
48	3.7174439D 02	2.4028558D-02	4.6468049D 02	1.9222847D-02	5.5761659D 02	1.6019039D-02
49	3.7620395D 02	2.3829516D-02	4.7025494D 02	1.9063613D-02	5.6430592D 02	1.5886344D-02
50	3.8060296D 02	2.3636953D-02	4.7575370D 02	1.8909562D-02	5.7090443D 02	1.5757968D-02